Quantum Field Theory

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November 16, 2014

Homework set 4, Due Wed Dec 3

1. Dirac equation for electron moving in the electromagnetic field can be obtained from the free Dirac equation by the replacement $i\partial_\mu \longrightarrow i\partial_\mu - eA_\mu$,

$$[\gamma^\mu (i\partial_\mu - eA_\mu) - m]\, \psi\left(\vec{x}, t\right) = 0$$

Then the equation for the positron is

$$[\gamma^\mu (i\partial_\mu + eA_\mu) - m]\, \psi_c\left(\vec{x}, t\right) = 0$$

Assume that ψ_c is related to ψ by

$$\psi_c = \tilde{C}\psi^*$$

$\tilde{C}$ is called the charge conjugation matrix.

   (a) Find $\tilde{C}$ in terms of Dirac γ matrices.

   (b) For the $v-$spinor of the form,

$$v(p,s) = N \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \\ 1 \end{pmatrix} \chi_s$$

   Compute its charge conjugate $v_c(p,s) = \tilde{C} v^*(p,s)$

   (c) To implement the charge conjugation for the fermion field, we write

$$\psi_c = C\psi C^{-1} = \tilde{C}\psi^*$$

   where C is the charge conjugation operator. Find the relation between $\bar{\psi}_c \gamma^\mu \psi_c$ and $\bar{\psi}\gamma^\mu\psi$.

2. Consider a free scalar field $\phi(x)$ where the 4-momentum operator is of the form,

$$P^\mu = \int d^3k\, k^\mu a^\dagger(k)\, a(k)$$

   (a) As a useful tool, show that for two operators A and B, the following identity holds

$$e^A B e^{-A} = B + [A,B] + \frac{1}{2}[A,[A,B]] + \frac{1}{3!}[A,[A,[A,B]]] + \cdots$$

   (b) Use this identity to show that

$$e^{iP\cdot x} a(k)\, e^{-iP\cdot x} = a(k)\, e^{-ik\cdot x}$$

   and

$$[P^\mu, \phi(x)] = i\partial^\mu \phi(x)$$

   (c) Let $|K\rangle$ be an eigenstate of P^μ, satisfying $P^\mu |K\rangle = K^\mu |K\rangle$. Show that

$$\langle K |\phi(x)\,\phi(y)| K\rangle = \langle K |\phi(x-y)\,\phi(0)| K\rangle$$

3. The propagator for a massless scalar field can be written in the form,

$$\Delta_F(x) = \int \frac{d^4x}{(2\pi)^4} \frac{e^{ik\cdot x}}{k^2 + i\varepsilon}$$

Carrying out the integration to show that

$$\Delta_F(x) = \frac{i}{4\pi} \frac{1}{x^2 - i\varepsilon}$$

4. In thequantization of free electromagnetic fields the mode expansion is of the form,

$$\overrightarrow{A}(\vec{x}, t) = \int \frac{d^3k}{\sqrt{2\omega(2\pi)^3}} \sum_\lambda \overrightarrow{\epsilon}(\vec{k}, \lambda)[a(k, \lambda)e^{-ikx} + a^\dagger(k, \lambda)e^{ikx}] \qquad w = k_0 = |\overrightarrow{k}|$$

where

$$\vec{\epsilon}(k, \lambda), \lambda = 1, 2 \quad \text{with } \vec{k} \cdot \vec{\epsilon}(k, \lambda) = 0$$

The quantization condition is of the form,

$$[\partial_0 A_i(\overrightarrow{x}, t),\ A_j(\overrightarrow{x}', t)] = -i\delta_{ij}^{tr}\delta_{ij}^{tr}(x - x')$$

Solve for $a(k, \lambda)$ and $a^+(k, \lambda)$ and compute the commutator,

$$[a(k, \lambda),\ a^\dagger(k', \lambda')]$$